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Valuations on Complemented Lattices

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It is proved that the space of all bounded real-valued valuations μ with $\mu(0) = 0$ on a complemented lattice is isomorphic to the space of all real-valued totally additive measures on a suitable complete Boolean algebra.

0. INTRODUCTION

In this paper we establish a natural isomorphism between the space of all bounded real-valued valuations μ with $\mu(0) = 0$ on a complemented lattice L and the space of all real-valued totally additive measures on $C(\tilde{L})$; here $\tilde{L}$ is a suitable completion of a quotient of L and $C(\tilde{L})$ the center of $\tilde{L}$; see Theorem 4.3. In particular, this answers in the affirmative the question of P. Pták of whether the space of all positive valuations μ with $\mu(0) = 0$ on an orthomodular lattice is isomorphic to a space of measures on a Boolean algebra.

A result—similar to that of this paper for *valuations*—was obtained in Weber (n.d.-a) for *exhaustive lattice uniformities* on orthomodular lattices: The lattice of all exhaustive lattice uniformities on L is isomorphic to the space of all order continuous FN-topologies on $C(\tilde{L})$. The analogous result can be derived from Avallone and Weber (n.d.) for a complemented modular lattice L. The proof of our main result (Theorem 4.3) is based on the idea which underlies the mentioned theorem from Weber (n.d.-a) for lattice uniformities, the Hahn-decomposition theorem (Section 2), and an extension theorem (Section 3) for valuations.

1. PRELIMINARIES

Throughout let L be a lattice with smallest element 0 *and greatest element* 1.

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We denote by $\Delta := \{(x, x): x \in L\}$ the diagonal of $L \times L$ and by $\mathbb{N}$ and $\mathbb{R}$ the sets of natural and real numbers, respectively.

A *valuation* on L is a function $\mu: L \to \mathbb{R}$ satisfying

$$\mu(x \vee y) + \mu(x \wedge y) = \mu(x) + \mu(y) \qquad \text{for all} \quad x, y \in L$$

μ is called *order continuous* (or *σ-order continuous*) if $(\mu(x_\gamma))$ converges to $\mu(x)$ whenever (x_γ) is, respectively, a monotone net (or a monotone sequence) order-converging to an element $x \in L$. It follows, e.g., from Fleischer and Traynor (1982), Theorem 3, that

$$N(\mu) := \{(x, y) \in L^2: \mu \text{ is constant on } [x \wedge y, x \vee y]\}$$

is a congruence relation for any valuation $\mu: L \to \mathbb{R}$.

1.1. Proposition. Let $(\mu_\alpha)_{\alpha \in A}$ be a family of valuations on L and $N = \bigcap_{\alpha \in A} N(\mu_\alpha)$. Then the quotient $\hat{L} := L/N$ is a modular lattice. Moreover, $\hat{L}$ is relatively complemented if L is complemented.

Proof. $\hat{L}$ is modular by Fleischer and Traynor (1982), Theorem 1, applied to the function $m: L \to \mathbb{R}^A$ defined by $m(x) := (\mu_\alpha(x))_{\alpha \in A}$. [Another proof of the modularity of $\hat{L}$ can be obtained by a modification of the second proof of Birkhoff (1984), Theorem X.2, that is near at hand.] Obviously, $\hat{L}$ is complemented. But any complemented modular lattice is relatively complemented by Birkhoff (1984), Theorem I.14.

1.2. Proposition. Let L be complemented and $\mu: L \to \mathbb{R}$ a valuation with $\mu(0) = 0$.

(a) If $\mu(x) \geq 0$ for all $x \in L$, then μ is increasing, hence bounded.

(b) If μ is bounded, then μ is of bounded variation, hence the difference of two increasing valuations.

Proof. For relative complemented lattices with 0 and 1, the assertion is formulated in Exercise 5 on p. 241 of Birkhoff (1984). One can reduce the assertion to the relative complemented case passing to the quotient $\hat{L} := L/N(\mu)$ defining on $\hat{L}$ a valuation $\hat{\mu}$ by $\hat{\mu}(\hat{x}) = \mu(x)$ for $x \in \hat{x} \in \hat{L}$. The quotient $\hat{L}$ is relatively complemented, by Proposition 1.1.

The *center* $C(L)$ of L is the set of elements of L which have one component 1_1 and the other 0_2, under some direct factorization $L \simeq L_1 \times L_2$ (Birkhoff, 1984, p. 67; Maeda and Maeda, 1970, p. 18). $C(L)$ is a Boolean sublattice of L (Maeda and Maeda, 1970, (4.15)).

For $\mu: L \to \mathbb{R}$ and $a \in L$, we define $\mu_a: L \to \mathbb{R}$ by $\mu_a(x) := \mu(a \wedge x)$.

1.3. Proposition. Let $\mu: L \to \mathbb{R}$ be a valuation with $\mu(0) = 0$, $a \in C(L)$, and a' its unique complement (Maeda and Maeda, 1970, (4.14)). Then μ_a,

$\mu_{a'}$ are valuations and $\mu = \mu_a + \mu_{a'}$; moreover, μ_a and $\mu_{a'}$ are, respectively, bounded or σ-order continuous or order continuous if μ is.

Proof. We may assume that $L = L_1 \times L_2$, $a = (1, 0)$, $a' = (0, 1)$. The assertion follows from the fact that then $\mu_a = \mu \circ p_1$ and $\mu_{a'} = \mu \circ p_2$, where p_1 and p_2 denote the projections from $L_1 \times L_2$ onto $L_1 \times \{0\}$ and $\{0\} \times L_2$, respectively.

A uniformity u on L which makes the lattice operations $\vee$ and $\wedge$ uniformly continuous is called a *lattice uniformity*. u is called *exhaustive* if every monotone sequence is Cauchy in (L, u). We denote by $N(u)$ the closure of the diagonal Δ with respect to u; then $N(u)$ is a congruence relation.

If $\mu: L \to \mathbb{R}$ is an increasing (= isotone) valuation, then $d_\mu(x, y) := \mu(x \vee y) - \mu(x \wedge y)$ defines a semimetric on L which induces an exhaustive lattice uniformity (Birkhoff, 1984, Theorems X.1 and X.4). μ is uniformly continuous with respect to d_μ. Moreover, $N(\mu) = N(d_\mu\text{-uniformity})$.

2. THE HAHN DECOMPOSITION

The proof of the injectivity of the isomorphism established in Theorem 4.1 is based on the following theorem.

2.1. Theorem. Let L be a complemented lattice, $\mu: L \to \mathbb{R}$ a valuation with $\mu(0) = 0$, and $\hat{L} = L/N(\mu)$.

(a) Then the following conditions are equivalent:

(1) There exists an element $a \in L$ with $\mu(a) = \sup \mu(L)$.

(2) There exists an element $b \in L$ with $\mu(b) = \inf \mu(L)$.

(3) There exist elements $a, b \in L$ such that μ_a and μ_b are valuations, $\mu_a(x) \geq 0$ and $\mu_b(x) \leq 0$ for $x \in L$, and $\mu = \mu_a + \mu_b$.

(b) If the conditions of (a) are satisfied, the elements a, b are uniquely determined except for equivalence with respect to $N(\mu)$. Moreover, the equivalence classes $\hat{a}$ and $\hat{b}$ containing a and b, respectively, belong to $C(\hat{L})$ and $\hat{b}$ is the complement of $\hat{a}$.

(c) If L is σ-complete and μ σ-order continuous, then the conditions of (a) are satisfied.

Proof. Part (c). By Weber (n.d.-b, 1.2.3), any σ-order continuous valuation on a σ-complete lattice attains its supremum.

In the proof of (a), (b) we may assume—passing to the quotient $\hat{L}$—that $N(\mu) = \Delta$. Then L is relatively complemented, by Proposition 1.1.

(i) μ attains in at most one point its supremum: Let $a_1, a_2 \in L$ with $\mu(a_1) = \mu(a_2) = \sup \mu(L) =: s$ and $a_1 \wedge a_2 \leq x \leq a_1 \vee a_2$. If x' is a relative complement of x in $[a_1 \wedge a_2, a_1 \vee a_2]$, then

$$\mu(x) + \mu(x') = \mu(x \wedge x') + \mu(x \vee x') = \mu(a_1 \wedge a_2) + \mu(a_1 \vee a_2)$$
$$= \mu(a_1) + \mu(a_2) = 2s$$

hence $\mu(x) = s = \mu(a_1)$. It follows that $(a_1, a_2) \in N(\mu) = \Delta$, i.e., $a_1 = a_2$.

(ii) Applying (i) to $-\mu$, one obtains that μ attains its infimum in at most one point.

(iii) Let a, b be elements according to (3). Then for any $x \in L$ we get, using Proposition 1.2(a), $\mu(x) = \mu_a(x) + \mu_b(x) \leq \mu_a(x) \leq \mu_a(1) = \mu(a)$. It follows that $\mu(a) = \sup \mu(L)$. Similarly, $\mu(b) = \inf \mu(L)$. By (i) and (ii), a and b are uniquely determined.

(iv) Assume that (1) or (2) is satisfied. We may assume that $\mu(a) = \sup \mu(L)$ for some $a \in L$; otherwise replace μ by $-\mu$. Let b be a complement of a. Then $\mu(b) = \inf \mu(L)$: In fact, if $x \in L$ and x' is a complement of x, then

$$\mu(x) = \mu(1) - \mu(x') \geq \mu(1) - \mu(a) = \mu(b)$$

It follows from (ii) that a has a unique complement. By Maeda and Maeda (1970), (4.20), an element of a relatively complemented lattice having a unique complement belongs to its center. Hence $a \in C(L)$ and therefore also $b \in C(L)$ (Maeda and Maeda, 1970, (4.14)). By Proposition 1.3, μ_a and μ_b are valuations and $\mu = \mu_a + \mu_b$. Let $x \in L$. If y is a relative complement of $a \wedge x$ in $[0, a]$, then $\mu_a(x) = \mu(a) - \mu(y) \geq 0$ and $\mu_b(x) = \mu(a \vee (b \wedge x)) - \mu(a) \leq 0$.

3. EXTENSION OF VALUATIONS

The proof of the surjectivity of the isomorphism established in Theorem 4.1 is based on the following Radon–Nikodym-type theorem. It can be easily deduced from the usual Radon–Nikodym theorem and the representation theorem (Sikorski, 1969, 29.1) for Boolean σ-algebras. But I prefer to give a direct proof repeating the idea for the proof of the Radon–Nikodym theorem.

3.1. Proposition. Let μ and ν be positive σ-additive measures on a σ-complete Boolean algebra A with $N(\nu) \subset N(\mu)$. Denote by S the set of measures of the type $\sum_{i=1}^{n} \alpha_i \nu_{d_i}$, where α_i are positive real numbers and $d_i \in A$. Then there is a sequence (μ_n) in S such that $\mu(x) = \sum_{n=1}^{\infty} \mu_n(x)$ for $x \in A$.

Proof. (i) Passing to the quotient $A/N(\nu)$, we may assume that ν is strictly positive. Let $\epsilon > 0$. We show that there exists a $\mu_0 \in S$ such that $\mu_0(x) \leq$

$\mu(x) \leq \mu_0(x) + 2\epsilon$ for $x \in A$. Let $\alpha > 0$; the Hahn decomposition theorem for the measure $\alpha\nu - \mu$ yields an element $x_\alpha \in A$ such that $\mu(x) \leq \alpha\nu(x)$ for $x \in [0, x_\alpha]$ and $\mu(x) \geq \alpha\nu(x)$ for $x \in [0, x'_\alpha]$, where x'_α denotes the complement of x_α. Since ν is strictly positive, $\alpha \leq \beta$ implies $x_\alpha \leq x_\beta$, and $x_n \uparrow 1$ (= unit of A). Choose $m \in \mathbb{N}$ with $\mu(x'_m) \leq \epsilon$ and $\nu(1)/m \leq \epsilon$. Put

$$x_0 := 0, \qquad d_i := x_{i/m} \backslash x_{i-1/m}, \qquad \mu_0 := \sum_{i=1}^{m^2} ((i - 1)/m)\nu_{d_i}$$

For $x \in A$ we have

$$((i - 1)/m)\nu(x \wedge d_i) \leq \mu(x \wedge d_i) \leq (i/m)\nu(x \wedge d_i)$$

Summing up these inequalities for $i = 1, \ldots, m^2$, one obtains

$$\mu_0(x) \leq \mu(x \wedge x_m) \leq \mu_0(x) + (1/m)\nu(x \wedge x_m)$$

It follows that

$$\begin{aligned}\mu_0(x) \leq \mu(x) &= \mu(x \wedge x_m) + \mu(x \backslash x_m)\\ &\leq \mu_0(x) + (1/m)\nu(1) + \mu(x'_m) \leq \mu_0(x) + 2\epsilon\end{aligned}$$

(ii) By (i), we can choose inductively $\mu_n \in S$ with

$$\mu_n(x) \leq \mu(x) - \sum_{i=1}^{n-1} \mu_i(x) \leq \mu_n(x) + (1/n) \quad \text{for} \quad n \in \mathbb{N} \text{ and } x \in A$$

Hence $\sum_{n=1}^{\infty} \mu_n(x) = \mu(x)$.

3.2. Corollary. Suppose that C is a σ-complete Boolean subalgebra of $C(L)$ and $a_n \downarrow 0$ in C implies $a_n \downarrow 0$ in L. Let μ_0: $C \to \mathbb{R}$ be a positive σ-additive measure and ν: $L \to \mathbb{R}$ a (σ-) order continuous increasing valuation such that $N(\nu \mid C) \subset N(\mu_0)$. Then there exists an increasing (σ-) order continuous valuation μ: $L \to \mathbb{R}$ extending μ_0.

Proof. Replacing ν by $\nu - \nu(0)$, we may assume that $\nu(0) = 0$; then $\nu \mid C$ is a σ-additive measure. Let S be the set of functions of the type $\sum_{i=1}^{n} \alpha_i\nu_{d_i}$, where α_i are positive real numbers and $d_i \in C$. All these functions are by Proposition 1.3 increasing (σ-) order continuous valuations on L. By Proposition 3.1, there exists a sequence (μ_n) in S such that $\sum_{n=1}^{\infty} \mu_n(x) = \mu_0(x)$ for $x \in C$. Denote by $\|\cdot\|_\infty$ the sup-norm. Since $\sum_{n=1}^{\infty} \|\mu_n\|_\infty = \sum_{n=1}^{\infty} \mu_n(1) = \mu_0(1) < +\infty$, the series $\sum_{n=1}^{\infty} \mu_n$ is uniformly convergent and its value $\mu := \sum_{n=1}^{\infty} \mu_n$ is an increasing (σ-) order continuous valuation extending μ_0.

3.3. Remark. Corollary 3.2 (as well Proposition 1.3 used in the proof of Corollary 3.2) also holds true if one replaces valuations by states on orthomodular lattices.

4. THE ISOMORPHY BETWEEN $bv(L)$ AND $ocv(C(\tilde{L}))$ FOR L COMPLEMENTED

We denote by $bv(L)$ and $ocv(L)$, respectively, the linear spaces of all bounded and all order continuous valuations $\mu: L \to \mathbb{R}$ with $\mu(0) = 0$. The space $bv(L)$ is a Banach space with respect to the sup-norm $\|\cdot\|_\infty$. If L is σ-complete, $ocv(L)$ is a closed linear subspace of $bv(L)$ (Weber, n.d.-b, 1.2.3). The spaces $bv(L)$ and $ocv(L)$ are partially ordered linear spaces with respect to the pointwise ordering.

4.1. Theorem. Let L be a complemented complete lattice such that $\bigcap_{\mu \in ocv(L)} N(\mu) = \Delta$.

(a) Then $C(L)$ is a complete sublattice of L; and $x_\alpha \downarrow x$ in $C(L)$ implies $x_\alpha \downarrow x$ in L (and dually).

(b) $\mu \mapsto \mu | C(L)$ defines an order-preserving isometry from $ocv(L)$ onto $ocv(C(L))$.

Proof. By Proposition 1.1, L is modular and relatively complemented. Therefore (a) follows from Grätzer (1978), Corollary III.4.11.

By (a), $r(\mu) := \mu | C(L) \in ocv(C(L))$ if $\mu \in ocv(L)$. Obviously, r is linear. Since, by Theorem 2.1, any $\mu \in ocv(L)$ attains its supremum and its infimum on $C(L)$, the map r preserves the sup-norm; in particular, r is injective.

Denote by V the set of all increasing valuations of $ocv(L)$. We now prove with the aid of Corollary 3.2 that any positive $\mu_0 \in ocv(C(L))$ can be extended to a valuation $\mu \in V$; this implies that r is surjective and order preserving. By Corollary 3.2 it is enough to show that $N(\mu_0) \supset N(\nu | C(L))$ for some $\nu \in V$.

Let $\nu \in V$. Then $\{x \in L: \nu(x) = 0\} = [0, a(\nu)]$ for some $a(\nu) \in L$. The congruence $N(\nu)$ is standard by Grätzer (1978), Theorem III.3.10, since L is a relatively complemented lattice with 0; hence $a(\nu)$ is a standard element by Grätzer (1978), Corollary III.3.3. By Grätzer (1978), Exercise 18, p. 144, and Maeda and Maeda (1970), (4.13), in a modular complemented lattice the center consists precisely of all standard elements. Hence $a(\nu) \in C(L)$ and $a(\nu)$ has, by Maeda and Maeda (1970), Remark (4.14), a unique complement $s(\nu)$ which also belongs to $C(L)$.

We show that $\sup_{\nu \in V} s(\nu) = 1$: For $s := \sup_{\mu \in V} s(\mu)$ and $\nu \in V$ we have $(s, 1) \in N(\nu)$, since $(s(\nu), 1) \in N(\nu)$. It follows with the aid of the Hahn decomposition Theorem 2.1 that $(s, 1) \in \bigcap_{\nu \in V} N(\nu) = \bigcap_{\nu \in ocv(L)} N(\nu) = \Delta$, hence $s = 1$.

Let $\mu_0 \in ocv(C(L))$ be a positive measure, $a \in C(L)$ with $\{x \in C(L): \mu_0(x) = 0\} = [0, a] \cap C(L)$ and s the complement of a in $C(L)$. Since the Boolean algebra $[0, s] \cap C(L)$ satisfies the countable chain condition and $\sup_{\nu \in V} s(\nu) = 1$, there is a sequence (ν_n) in V with $s \leq \sup_{n \in \mathbb{N}} s(\nu_n)$ and therefore a sequence (a_n) of pairwise disjoint elements of $C(L)$ with $s = \sup_{n \in \mathbb{N}} a_n$ and $a_n \le s(\nu_n)$ for $n \in \mathbb{N}$. Choose $\epsilon_n > 0$ with $\epsilon_n \nu_n(1) \le 2^{-n}$; put $\lambda_n(x) := \nu_n(x \wedge a_n)$ for $x \in L$. Then $\nu := \sum_{n=1}^{\infty} \epsilon_n \lambda_n \in V$ and $s(\nu) = s$, i.e., $N(\nu \mid C(L)) = N(\mu_0)$.

Our main theorem, Theorem 4.3, is a consequence of the algebraic result, Theorem 4.1, and the following topological result.

4.2. Proposition. Let w be a Hausdorff lattice uniformity on L and $(\tilde{L}, \tilde{w})$ its uniform completion.

(a) Then $\tilde{L}$ becomes a lattice by continuously extending the lattice operations from L to $\tilde{L}$.

(b) If w is exhaustive, then $\tilde{w}$ is order continuous and $(\tilde{L}, \le)$ is a complete lattice.

(c) If w is exhaustive and L a complemented modular lattice, then also $\tilde{L}$ is complemented and modular.

For (a), (b) see Kiseleva (1967) or Weber (1991), 1.3.1 and 6.15. (c) is proved in Weber (n.d.-c).

We use in the following result the fact that for any lattice uniformity w on L the quotient uniformity $\hat{w}$ of w on $L/N(w)$ is a Hausdorff lattice uniformity (Weber, 1991, 1.2.4).

4.3. Theorem. Let L be a complemented lattice and w the (exhaustive) lattice uniformity generated by the family of all semimetrics d_μ defined by increasing valuations μ: $L \to \mathbb{R}$. Let $\hat{w}$ be the quotient uniformity of w on $\hat{L} := L/N(w)$ and $(\tilde{L}, \tilde{w})$ the completion of $(\hat{L}, \hat{w})$. Then the spaces $bv(L)$, $bv(\hat{L})$, $ocv(\tilde{L})$, $ocv(C(\tilde{L}))$ are isometric Banach spaces and isomorphic as Riesz spaces. The natural order-preserving linear isometries are defined as follows:

$$bv(L) \ni \mu \mapsto \hat{\mu} \in bv(\hat{L}), \text{ where } \hat{\mu}(\hat{x}) = \mu(x) \text{ and } x \in \hat{x} \in \hat{L}$$
$$ocv(\tilde{L}) \ni \tilde{\mu} \mapsto \tilde{\mu} \mid \hat{L} \in bv(\hat{L})$$
$$ocv(\tilde{L}) \ni \tilde{\mu} \mapsto \tilde{\mu} \mid C(\tilde{L}) \in ocv(C(\tilde{L}))$$

Proof. Obviously, the first map ($\mu \mapsto \hat{\mu}$) is an order-preserving linear isometry. From the facts that any $\tilde{\mu} \in ocv(\tilde{L})$ is bounded, by Weber (n.d.-b), 1.2.3 [or by Theorem 2.1, Proposition 4.2(c)], and that the continuous extension of any $\hat{\mu} \in bv(\hat{L})$ on $(\tilde{L}, \tilde{w})$ belongs to $ocv(\tilde{L})$, since $\tilde{w}$ is order continuous by Proposition 4.2(b), it easily follows that the second map ($\tilde{\mu} \mapsto \tilde{\mu} \mid L$) is an order-preserving linear isometry. The third map ($\tilde{\mu} \mapsto \tilde{\mu} \mid C(\tilde{L})$) is by Theorem 4.1 an order-preserving linear isometry, since $\tilde{L}$ is complete by Proposition 4.2(b) and complemented by Proposition 4.2(c) and Proposition 1.1.

The space $ocv(C(\tilde{L}))$ coincides with the space of all real-valued totally additive measures on $C(\tilde{L})$.

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